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Disjunctivity and Alternativity in Projection Logics[†]

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We study the notions of disjunctivity and alternativity of orthomodular posets in the context of orthoprojections or skew projections in C^*-algebras.

1. INTRODUCTION

Projections are of crucial importance in the study of operator algebras (e.g., Blackadar, 1994). They also serve to form domains for measures and signed measures in the noncommutative measure and integration theory.

In this paper, we examine orthomodular posets (OMPs) of the orthoprojections or of all idempotents (= "skew projections") in C^*-algebras and try to investigate whether these are disjunctive or alternative. The concepts of disjunctivity and alternativity have arisen in the general theory of OMPs owing to Godowski (1979) and Ovchinnikov (1994). The present paper is an attempt to study abstract conditions that may be imposed on an arbitrary OMP in the particular case of projection logics of operator algebras and to thereby study the algebras themselves.

Basic notions of the theory of OMPs and C^*-algebras may be found in Kalmbach (1983) and Murphy (1990).

In Section 2, we present an example of a C^*-algebra whose OMPs of all orthoprojections and of all skew projections are atomic and nonatomistic. Thus the OMPs are nondisjunctive and, in particular, nonalternative.

In Section 3, we show that the OMP of all skew projections in every von Neumann algebra is disjunctive (though need not be alternative).

In Section 4, we pose several open problems.

[†] This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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2. NONDISJUNCTIVITY IN C^*-ALGEBRAS

Definition 2.1. An OMP E is called *disjunctive* (Godowski, 1979) if

$$x, y \in E, \quad x \nleq y \Rightarrow \exists c \in E \backslash \{0\} \qquad (c \leq x, c \wedge y = 0) \tag{1}$$

Definition 2.2. An OMP E is called *alternative* (Ovchinnikov, 1994) (or *weakly orthocomplete*, as some authors have begun to say; see, e.g., de Lucia and De Simone, 1998) if every orthogonal subset of E either has a supremum or has no minimal majorants.

Definition 2.3 (Kalmbach, 1983). An OMP E is called *atomic* if every nonzero element in E majorizes an atom in E.

Definition 2.4 (Kalmbach, 1983). An OMP E is called *atomistic* if every element in E is the supremum of a set of atoms.

It is known that every alternative OMP is disjunctive and every atomic and disjunctive OMP is atomistic (Ovchinnikov, 1994). Therefore, the following is an example of two nondisjunctive OMPs.

Example 2.5. We aim to construct a C^*-algebra $\mathcal{M}$ such that the OMP $\pi(\mathcal{M})$ of all orthoprojections and the OMP $\mathfrak{B}(\mathcal{M})$ of all skew projections in $\mathcal{M}$ are atomic and nonatomistic.

Consider the compact set $T = \{-1, -1/2, -1/3, \ldots\} \cup [0, 1]$ and the C^*-algebra $C(T, M_2)$ of all continuous functions on T with values in the C^*-algebra M_2 of all complex 2×2-matrices with respect to the pointwise algebraic operations and the sup-norm.

Every element in $\Pi(C(T, M_2))$ is a continuous function $P(\cdot)$ on T such that every $P(t)$ is an orthogonal projection. Let $P(\cdot) \in \Pi(C(T, M_2))$ and $P(0) \neq 0$. As P is continuous, $P(t) \neq 0$ for each t in a neighborhood of 0. As the connected component of 0 in $\Pi(C(T, M_2))$ is $\{0\}$, $P(t) \neq 0$ for all $t \in [0, 1]$. Thus, an atom in $\Pi(C(T, M_2))$ is just a function of the form $I_{\{x\}}\pi$, where $x \in \{-1, -1/2, -1/3, \ldots\}$ and π is a one-dimensional orthoprojection, and $\Pi(C(T, M_2))$ is atomic. Consider $P_1(t) = \pi$ $(t \in T)$, where π is a one-dimensional orthoprojection in M_2. The atoms majorized by $P_1(\cdot)$ are $I_{\{x\}}\pi$, where $x \in \{-1, -1/2, -1/3, \ldots\}$. For every $t \in [0, 1]$, let $U(t)$ be the rotation through the angle t in $\mathbb{C}^2$. Put

$$P_2(t) = \begin{cases} \pi & \text{if} \quad t \in \{-1, -1/2, -1/3, \ldots\} \\ U(t)^{-1}\pi U(t) & \text{if} \quad t \in [0, 1] \end{cases} \qquad (t \in T)$$

It is easy to see that $P_1(\cdot)$ and $P_2(\cdot)$ are two different minimal majorants of the aforementioned set of atoms. Thus $\Pi(C(T, M_2))$ is nonatomistic.

The case of $\mathfrak{B}(C(T, M_2))$ is similar.

3. DISJUNCTIVITY IN VON NEUMANN ALGEBRAS

Note that an arbitrary skew projection P in $B(H)$, H being a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H, is defined by $P = \{E, F\}$, where E and F are closed subspaces of H, $E + F = H$, and $E \cap F = \{0\}$. Here, $E = \text{Im } P$ and $F = \text{Ker } P$.

Theorem 3.1. If $\mathcal{M}$ is a von Neumann algebra acting in a complex Hilbert space H, then the OMP $\mathfrak{V}(\mathcal{M})$ is disjunctive.

Proof. We use the following simple facts:

(a) for every $T \in \mathcal{M}$, the orthogonal projection onto $\overline{\text{Im } T}$ belongs to $\mathcal{M}$.

(b) if A is an invertible positive operator in $\mathcal{M}$, then the inner product $(x, y)_A := (Ax, y)$ defines a new involution $T \to T^{*A}$ onto $B(H)$ which maps $\mathcal{M}$ into $\mathcal{M}$. The $\mathcal{M}$ with this involution becomes a von Neumann algebra, say $\mathcal{M}_A$, in the Hilbert space $(H, (\cdot, \cdot)_A)$.

(c) any skew projection P in $\mathcal{M}$ is an orthoprojection in $\mathcal{M}_A$, where (Mushtari, 1989, 1998) $A = P^*P + (I - P^*)(I - P)$.

Consider two elements $P = \{E, F\}$ and $P_1 = \{E_1, F_1\}$ in $\mathfrak{V}(\mathcal{M})$ such that $P \not\leq P_1$. Put $A = P^*P + (I - P^*)(I - P)$. We denote by $\ominus_A$ the orthogonal difference with respect to $(\cdot, \cdot)_A$.

The following two cases are possible:

1. $E \not\subset E_1$. In this case, we denote $G = E \cap E_1$ and consider $E_2 = E \ominus_A G$. Define a skew projection $P_2 \in \mathcal{M}_A$ by $\text{Im } P_2 = E_2$. Then $P_2 \leq P$ because $E_2 \subset E$ and P, P_2 are Hermitian with respect to the same $(\cdot, \cdot)_A$, and $P_1 \wedge P_2 = 0$ because $E_2 \cap E_1 = \{0\}$.

2. $E \subset E_1$, $F \not\supset F_1$. If $P \wedge P_1 = 0$, then (1) is fulfilled (with $x = P$, $y = P_1$, $c = P$). If $P \wedge P_1 \neq 0$, then $\overline{\text{Lin} F \cup F_1} \neq H$. We put $G = H \ominus_A \overline{\text{Lin} F \cup F_1}$, $F_2 = \overline{\text{Lin} F \cup G}$, and $E_2 = H \ominus_A F_2$. Since $F_2 \supset F$, we have $P_2 = \{E_2, F_2\} \leq P$. Since $\overline{\text{Lin} F_1 \cup F_2} = H$, we have $P_2 \wedge P_1 = 0$. Since $F \cap F_1 \neq F$, it follows that $F_2 \neq H$, whence $P_2 \neq 0$.

Theorem 3.2. The OMP $\mathfrak{V}(B(H))$, H being a separable infinite-dimensional Hilbert space, is nonalternative.

Proof. Let us decompose H as $H = \overline{\text{Lin}\{x_i\}} \oplus \overline{\text{Lin}\{y_i\}} \oplus \overline{\text{Lin}\{z_i\}}$, where $\{x_i\} \cup \{y_i\} \cup \{z_i\}$ is an orthonormal basis and $\{x_i\}$, $\{y_i\}$, $\{z_i\}$ are infinite sequences. Suppose also that

(i) $\{h_i\}$ is a linearly independent system whose linear hull is dense in $\overline{\text{Lin}\{y_i\}} \oplus \overline{\text{Lin}\{z_i\}}$.

(ii) For any n,

$$\text{Lin}\{h_i: i \leq n\} \cap \overline{\text{Lin}(\{y_i\} \cup\text{er } \{z_i: i > n\})} = \{0\}$$

We propose the following construction of $\{h_i\}$:

$$h_{2n-1} = z_n + \frac{1}{2^{n+3}} z_{2n-1}, \qquad h_{2n} = y_n + \frac{1}{2^{n+3}} z_{2n}, \qquad \forall n$$

We have $\Sigma_n \|z_n - h_{2n-1}\| + \Sigma_n \|y_n - h_{2n}\| < 1/2$. By the Krein–Milman–Rutman theorem (Lindenstrauss and Tzafriri, 1977, Proposition 1.a.9), $\{h_n\}$ is a Schauder basis in $\overline{\mathrm{Lin}\{y_i\}} \oplus \overline{\mathrm{Lin}\{z_i\}}$. This proves (i). The property (ii) can be readily verified.

It follows from (i) and (ii) that the following couples define continuous projections:

$$P_1 = \{\mathrm{Lin}\{h_1\}, \overline{\mathrm{Lin}\{\{x_i\} \cup \{y_i\} \cup \{z_i\} \backslash \{z_1\}\}}\}$$
$$P_2 = \{\mathrm{Lin}\{h_1, h_2\}, \overline{\mathrm{Lin}\{\{x_i\} \cup \{y_i\} \cup \{z_i\} \backslash \{z_1, z_2\}\}}\}$$
$$\vdots$$

Now, we will prove that this sequence possesses no supremum. By construction, every majorant of the sequence is a projection $Q = \{M, K\}$ satisfying $M \supset E$ and $K \subset F$, where

$$E = \overline{\mathrm{Lin}\{\{y_i\}\} \cup \{z_i\}}, \qquad F = \overline{\mathrm{Lin}\{\{x_i\} \cup \{y_i\}}\}$$

If $Q = \{M, K\}$ is a supremum of our sequence, then $M = E$. Suppose on the contrary that E is a proper subspace of M. Let us consider a closed hyperplane M_1 in M containing E. The $M_1 + K$ is a closed hyperplane in H; therefore, $M_1 + K \not\supset F$ (otherwise, $M_1 + K \supset M_1 + F \supset E + F = H$). Thus there exists a nonzero $x \in F \backslash (M_1 + K)$. It is easy to see that the couple $\{M_1, \mathrm{Lin}\,(\{x\} \cup K)\}$ defines a skew projection. Actually, $M_1 + \mathrm{Lin}(\{x\} \cup K) \neq M_1 + K$. Since $M_1 + K$ is a hyperplane, $M_1 + \mathrm{Lin}(\{x\} \cup K) = H$. The couple $\{M_1, \mathrm{Lin}(\{x\} \cup K)\}$ is majorized by $\{M, K\}$ and, at the same time, majorizes all P_n. This contradicts the choice of $\{M, K\}$.

So, $M = E$. Any minimal majorant has the form $\{E, K\}$, where $K \subset F$ and vice versa. Obviously there exist incomparable majorants, $\{E, K_1\}$ and $\{E, K_2\}$, for example, with $K_1 = \overline{\mathrm{Lin}\{x_i\}}$ and $K_2 = \overline{\mathrm{Lin}\{x_i + y_i\}}$.

4. OPEN QUESTIONS

Problem 4.1. Describe the C^*-algebras whose OMP of all skew projections (a version: of all orthoprojections) is (a) disjunctive, (b) alternative.

Next, we need the following strengthening of the notion of an atomistic OMP.

Definition 4.2. An OMP is called *orthoatomistic* (Ovchinnikov, 1994) if its every element is the supremum of an orthogonal set of atoms.

Ovchinnikov (1994) gave an example of an atomistic nonorthoatomistic OMP that was an uncountable concrete logic.

Problem 4.3. Does there exist a countable atomistic nonorthoatomistic OMP?

Problem 4.4. Does there exist a C^*-algebra whose OMP of all orthoprojections (a version: skew projections) is atomistic and nonorthoatomistic?

Finally, observe that every atomistic nonorthoatomistic OMP is necessarily disjunctive and nonalternative (Ovchinnikov, 1994).

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